

Signature Change and Clifford Algebras¹

David Miralles,² Josep Manel Parra,² and Jayme Vaz, Jr.³

Received August 27, 1999; revised March 28, 2000

Given the real Clifford algebra of a quadratic space with a given signature, we define a new product in this structure such that it simulates the Clifford product of a quadratic space with another signature different from the original one. Among the possible applications of this new product, we use it in order to write the Minkowskian Dirac equation over the Euclidean spacetime and to define a new duality operation in terms of which one can find self-dual and anti-self-dual solutions of gauge fields over Minkowski spacetime analogous to the ones over Euclidean spacetime and without needing to complexify the original real algebra.

1. INTRODUCTION

Applications of Clifford algebras (CA) can be found in Baylis (1996), Chisholm *et al.* (1986), Delanghe *et al.* (1993), Dietrich *et al.* (1998), and Micali *et al.* (1992). In order to define the CA of a given vector space V , one needs to endow V with a symmetric bilinear form $g_{p,q}$. The structure of the CA depends also on the signature of g . Real Clifford algebras $Cl_{p,q}$ and $Cl_{p',q'}$ ($p + q = p' + q' = n$) are in general not isomorphic, that is, when we change the signature, we get in general different Clifford algebras. For example, the Clifford algebras $Cl_{1,3}$ and $Cl_{3,1}$ are not isomorphic; indeed, in terms of matrix algebras, the former is isomorphic to the algebra of 2×2 quaternionic matrices, while the latter is isomorphic to the algebra of 4×4 real matrices.

¹D.M. is grateful to Departamento de Matemática Aplicada, Universidade Estadual de Campinas (UNICAMP), for financial support during the preparation of this work. J.M.P. acknowledges support from the Spanish Ministry of Education contract No. PB96-0384 and the Institut d'Estudis Catalans.

²Departament de Física Fonamental, Universitat de Barcelona, Barcelona, Catalonia, Spain; e-mail: david@ffn.ub.es

³Departamento de Matemática Aplicada, Universidade Estadual de Campinas, 13081-970 Campinas, SP, Brasil; e-mail: vaz@ime.unicamp.br

The Euclidean formulation of field theories is a fundamental tool in modern physics (Glimm and Jaffe, 1987). Sometimes it seems to be even crucial, as in the theory of instantons, in finite-temperature field theory, and in lattice gauge theory. Going from a Euclidean to a Minkowskian theory or vice versa involves changing the signature of the metric over the spacetime, and in general a Minkowskian theory is transformed in a Euclidean theory by analytical continuation, that is, by making $t \rightarrow it$. The interpretation of making $t \rightarrow it$ is not trivial; see van Nieuwenhuizen and Waldron (1996), where it was interpreted as a rotation in a five-dimensional spacetime.

Despite the ingenuity of an approach like that of van Nieuwenhuizen and Waldron (1996) in interpreting $t \rightarrow it$ as a rotation in a five-dimensional space, we believe it is unsatisfactory since the use of an additional time coordinate in spacetime appears to us to be meaningless. Indeed, it would be much more satisfactory if one could find an approach to describe the signature change where there is no need to introduce extra dimensions. In the case where the problem involves CA, the situation is even more problematic since the signature-changed CA and the original one may not be isomorphic. A possible way to overcome this situation is to complexify the original real CA since the structure of complex CA depends only on the dimension of V , but this approach can also be seen as a result of introducing an extra dimension (see below).

The objective of this paper is to introduce an *algebraic* approach to the change of signature in CA where there is *no need to introduce any extra dimension* to describe it. The signature change appears as a transformation on the algebraic structure underlying the theory. The idea is to propose an operation that “simulates” the product properties of the signature-changed space in terms of the original space or vice versa. The particular case we have in mind is the one involving the four-dimensional spacetime, where we introduce an operation “simulating” the properties of Minkowski spacetime in terms of a Euclidean spacetime and vice versa. This operation will be called “vee product” (since it will be denoted by a \vee in order to distinguish it from the usual Clifford product that will be denoted by juxtaposition). The advantage of this approach is obvious since we can retain the “physics” (the Minkowskian properties) in a suitable mathematical world (the Euclidean spacetime). The fact that we can define the vee product in terms of the Clifford product means that we can describe the Minkowskian properties in terms of Euclidean spacetime and vice versa.

Our approach has been inspired by the work of Lounesto (1997). There are differences in the method and interpretation. Lounesto only discussed some problems involving the case of signature change corresponding to opposite signatures, that is, (p, q) and (q, p) . Cases like (p, q) and $(p + q, 0)$ were not considered by Lounesto and this is the case we have when considering

Minkowskian and Euclidean spacetimes. Some applications of this are discussed here; some others can be found in Pezzaglia and Adams (1997).

The version of Dirac equation we obtain by making $t \rightarrow it$ (we shall call it the Euclidean Dirac equation) has physical properties that are obviously different from the original Dirac equation (which we shall call the Minkowskian Dirac equation). The question we want to address in this context is whether it is possible to obtain a new equation that exhibits the same physical properties of the original equation. More specifically, our idea is to write the Minkowskian Dirac equation in the Euclidean spacetime, which should obviously be different from a Euclidean Dirac equation in a Euclidean spacetime. Minkowskian and Euclidean spacetimes are different worlds, both mathematically and physically speaking, and we want to simulate the Minkowskian scenario in a Euclidean world. The idea is to write an equation in Euclidean spacetime in terms of the vee product such that it is equivalent to the original equation in terms of the Clifford product in Minkowski spacetime. Of course the Dirac equation in terms of the vee product is expected to be different from the Euclidean Dirac equation.

The Dirac equation is in general formulated in terms of 4×4 complex matrices—the gamma matrices—obeying the relation $\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2g_{\mu\nu}$, where $g_{\mu\nu} = \text{diag}(1,1,1,1)$ in the Euclidean case and $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ in the Minkowskian case. This algebra is the matrix representation of the complex Clifford algebra $Cl_{\mathbb{C}}(4)$ (Porteous, 1995). On the other hand, this algebra is the complexification of the real algebras $Cl_{1,3}$ and $Cl_{4,0}$ associated with the Minkowskian and Euclidean spacetimes, respectively, that is, $Cl_{\mathbb{C}}(4) \simeq \mathbb{C} \otimes Cl_{1,3} \simeq \mathbb{C} \otimes Cl_{4,0}$. Moreover, $Cl_{\mathbb{C}}(4)$ is also isomorphic to the real algebra $Cl_{4,1}$, while the real algebras $Cl_{1,3}$ and $Cl_{4,0}$ —which are also isomorphic—are isomorphic to the even subalgebra of $Cl_{4,1}$ (Figueiredo *et al.*, 1990). However $Cl_{3,1}$ and $Cl_{4,0}$ are not isomorphic, and we shall see how to consider this case also. It is not the isomorphism between $Cl_{1,3}$ and $Cl_{4,0}$ that matters in this discussion. All these facts show that when we complexify the real algebra $Cl_{1,3}$ getting $\mathbb{C} \otimes Cl_{1,3} \simeq Cl_{\mathbb{C}}(4) \simeq Cl_{4,1}$, we are introducing an extra dimension to the Minkowskian spacetime such that the extra dimension is of the type of Euclidean time. In the same way, when we complexify the real algebra $Cl_{4,0}$ getting $\mathbb{C} \otimes Cl_{4,0} \simeq Cl_{\mathbb{C}}(4) \simeq Cl_{4,1}$, we are introducing an extra dimension to the Euclidean spacetime such that the extra dimension is of the type of Minkowskian time.

Our idea is to not use extra dimension, and *one certain way to do this* is to avoid using any of these complexified structures. This can be achieved using the real formulation of Dirac theory due to Hestenes (1966). One can easily formulate the Dirac theory in terms of the Clifford algebra $Cl_{1,3}$, which is isomorphic to the algebra of 2×2 quaternionic matrices. A Dirac spinor in this way is represented by a pair of quaternions, but we prefer to use the

form of Dirac equation called the Dirac–Hestenes equation in terms of the so-called Dirac–Hestenes spinor (Figueiredo *et al.*, 1990). The reason for our choice is simple. The Dirac–Hestenes spinor is represented by an element of the even subalgebra $Cl_{1,3}^+$, while a Dirac spinor is an element of an ideal of $Cl_{1,3}$. Both formulations are equivalent (Rodrigues *et al.*, 1996), but if we use the former one, we avoid the problem of considering the transformation between different ideals (in fact left and right ideals), which will happen in the latter case, and in order to not do unnecessary work—and even hidden some fundamental facts—we prefer to use the Dirac–Hestenes equation.

As another application, we discuss the problem of finding self-dual and anti-self-dual solutions of gauge fields. Since the group being Abelian or not is irrelevant for this matter, we shall restrict our attention to the Abelian case. As is well known, in the Minkowski spacetime there do not exist (real) solutions to the problem $*F = \pm F$, where F is the 2-form representing the electromagnetic field and $*$ is the Hodge star operator. However, in Euclidean spacetime, this problem has solutions and they are given by $E = \pm B$, where E and B are the electric and magnetic components of F . Now, using the operation we discussed above, we can define a new Hodge-like operator on Minkowski spacetime such that in relation to this new operator we have self-dual and anti-self-dual solutions for that problem on Minkowski spacetime. Moreover, using this Hodge-like operator, we can completely simulate a Euclidean metric while still working on Minkowski spacetime. We also show that the relation between those Hodge-like operators is given by the parity operation.

2. PRELIMINARIES

There are many different ways to define Clifford algebras (Gilbert and Murray, 1991). Our approach has been chosen due to the direct introduction to the Clifford product (Riesz, 1993).

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis for $\mathbb{R}^{p,q}$, where $\mathbb{R}^{p,q}$ is a real vector space of dimension $n = p + q$ endowed with an interior product $g: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \rightarrow \mathbb{R}$. Writing the quadratic form $\sum_{i=1}^n \sum_{j=1}^n g_{ij}x_i x_j$ as the square of the linear expression $\sum_{i=1}^n x_i \mathbf{e}_i$ and assuming the distributive property, we obtain the well-known expression for the Clifford algebra $Cl(\mathbb{R}^{p,q}, g) \equiv Cl_{p,q}$

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 2g_{ij} \quad (1)$$

where g_{ij} are the metric components. This defines the Clifford product, which has been denoted by juxtaposition.

The exterior product \wedge is associative, bilinear, and skew-symmetric on vectors. By applying it to our orthogonal basis, we can construct a new vector space $\Lambda^2(\mathbb{R}^{p,q})$ whose elements are called bivectors, i.e., $\wedge: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \rightarrow$

$\Lambda^2(\mathbb{R}^{p,q})$. The skew-symmetric property allows us to extend the definition to $\Lambda^n(\mathbb{R}^{p,q})$. In general,

$$\wedge: \Lambda^k(\mathbb{R}^{p,q}) \times \Lambda^l(\mathbb{R}^{p,q}) \rightarrow \Lambda^{k+l}(\mathbb{R}^{p,q})$$

If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^{p,q}$, then 1 and the Clifford products $\mathbf{e}_{i_1} \dots \mathbf{e}_{i_k}$ ($1 \leq i_1 < i_2 < \dots < i_k \leq n$) will establish a basis for $Cl_{p,q}$ which has dimension 2^n . If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthogonal basis, then $\mathbf{e}_1 \dots \mathbf{e}_n = \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_n$, which is usually called the volume element. It follows that $Cl_{p,q}$ and $\Lambda(\mathbb{R}^{p,q}) = \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^{p,q})$ are isomorphic as vector spaces. Therefore, a general element $A \in Cl_{p,q}$ takes the form

$$A = A_0 + A_1 + \dots + A_n = a_0 + a^i \mathbf{e}_i + a^{ij} \mathbf{e}_{ij} + \dots + a^{1\dots n} \mathbf{e}_{1\dots n} \in Cl_{p,q}$$

where A_r , called an r -vector, belongs to $\Lambda^r(\mathbb{R}^{p,q}) \subset Cl_{p,q}$ ($r = 0, 1, \dots, n$). It is convenient to define a set of projectors $\langle \rangle_r$ as $\langle \rangle_r: \Lambda(V) \rightarrow \Lambda^r(V)$, i.e., $\langle A \rangle_r = A_r$. An important property is that Clifford algebra is a \mathbb{Z}_2 -graded algebra, i.e., we can divide it into even ($Cl_{p,q}^+$) and odd ($Cl_{p,q}^-$) grades. $Cl_{p,q}^+$ is a subalgebra of $Cl_{p,q}$ called an even subalgebra. Some important identities that we will use later are ($a \in \mathbb{R}^{p,q}$, $B \in \Lambda^r(\mathbb{R}^{p,q}) \subset Cl_{p,q}$)

$$\begin{aligned} aB &= a \cdot B + a \wedge B \\ a \wedge B &\equiv \langle aB \rangle_{r+1} = \frac{1}{2}(aB + (-1)^r Ba) \\ a \cdot B &\equiv \langle aB \rangle_{r-1} = \frac{1}{2}(aB - (-1)^r Ba) \end{aligned}$$

3. THE VEE PRODUCT

Let V be a vector space of dimension $n = 4$. We have five different Clifford algebras depending on the signature: $Cl_{4,0}$, $Cl_{3,1}$, $Cl_{2,2}$, $Cl_{1,3}$, and $Cl_{0,4}$.

We shall consider the case involving the algebras $Cl_{1,3}$ and $Cl_{4,0}$. Let $A, B \in Cl_{4,0}$ and AB be its Clifford product. Now we define a new product, which we call a *vee* product, $A \vee B$, simulating the $Cl_{1,3}$ Clifford product in $Cl_{4,0}$. After the selection in $\mathbb{R}^{4,0}$ of an arbitrary unit vector \mathbf{e}_0 to represent the *fourth dimension* and the completion of the basis with three other orthonormal vectors \mathbf{e}_i , we define, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{4,0}$,

$$\mathbf{u} \vee \mathbf{v} := (-1)[\mathbf{v}\mathbf{u} - 2(\mathbf{v} \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{u})] \quad (2)$$

Using this product and the $Cl_{4,0}$ standard basis, it is easy to prove that $\mathbf{e}_0 \vee \mathbf{e}_0 = 1$ and $\mathbf{e}_i \vee \mathbf{e}_i = -1$ ($i = 1, 2, 3$), while $\mathbf{e}_\mu^2 = \mathbf{e}_\mu \mathbf{e}_\mu = 1$ ($\mu = 0, 1, 2, 3$). Moreover, for $\mathbf{u}, \mathbf{w} \in \mathbb{R}^{4,0}$,

$$\mathbf{u}\mathbf{w} + \mathbf{w}\mathbf{u} = 2\mathbf{u} \cdot \mathbf{w} = 2(u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3) \quad (3)$$

but if we use the *vee* product, then

$$\begin{aligned}\mathbf{u} \vee \mathbf{w} + \mathbf{w} \vee \mathbf{u} &= -\mathbf{w}\mathbf{u} + 2(\mathbf{w} \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{u}) - \mathbf{u}\mathbf{w} + 2(\mathbf{u} \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{w}) \\ &= 2(u_0w_0 - u_1w_1 - u_2w_2 - u_3w_3)\end{aligned}$$

A slightly more general case is when one has a vector and a k -graded element, i.e., \mathbf{v} and B_k ,

$$B_k \vee \mathbf{v} = (-1)^k[\mathbf{v}B_k - 2(\mathbf{v} \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot B_k)] \quad (4)$$

$$\mathbf{v} \vee B_k = (-1)^k[B_k\mathbf{v} - 2(B_k \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{v})] \quad (5)$$

One can see that \vee is associative, that is, $\mathbf{v} \vee (\mathbf{u} \vee \mathbf{w}) = (\mathbf{v} \vee \mathbf{u}) \vee \mathbf{w}$. Moreover, the vee product preserves the multivectorial structure since

$$\begin{aligned}\frac{1}{2}[\mathbf{u}, \mathbf{v}]_{\vee} &= \frac{1}{2}(\mathbf{u} \vee \mathbf{v} - \mathbf{v} \vee \mathbf{u}) = \frac{1}{2}(-\mathbf{v}\mathbf{u} + 2u_0v_0 + \mathbf{u}\mathbf{v} - 2u_0v_0) \\ &= \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}) = \frac{1}{2}[\mathbf{u}, \mathbf{v}] = \mathbf{u} \wedge \mathbf{v}\end{aligned}$$

Finally, we can generalize those expression as

$$A_l \vee B_k = (-1)^{kl}[B_kA_l - 2(B_k \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot A_l)] \quad (6)$$

4. DIRAC EQUATION AND VEE PRODUCT

We shall use the Dirac–Hestenes equation (Figueiredo *et al.*, 1990; Hestenes, 1966; Rodrigues *et al.*, 1996; Lounesto, 1993; Parra, 1992) in $Cl_{1,3}$, where ∇ denotes the Dirac operator,

$$\nabla\psi\gamma_{21} - m\psi\gamma_0 = 0, \quad \nabla = \gamma_0\partial_0 - \gamma_1\partial_1 - \gamma_2\partial_2 - \gamma_3\partial_3$$

and γ_μ ($\mu = 0, 1, 2, 3$) are interpreted as vectors in $Cl_{1,3}$ and $\psi = \psi(x) \in Cl_{1,3}^+$, $\forall x \in M$, where M is the Minkowskian manifold. Let ask a question: How can one simulate the Minkowskian Dirac equation in a Euclidean formulation? The answer is given for the vee product.

Multiplying on the right for γ_{12} , we can write

$$\nabla\psi - m\psi\gamma_{012} = 0 \quad (7)$$

Considering $\psi \in Cl_{4,0}^+$, $\forall x \in M$, and using \mathbf{e} -notation for the $Cl_{4,0}$ elements, we write the Dirac equation (7) in the Euclidean spacetime using the \vee product as

$$\nabla \vee \psi - m\psi \vee \mathbf{e}_{012} = 0 \quad (8)$$

where $\nabla = \mathbf{e}^\mu \partial_\mu$ with $\mathbf{e}^\mu = \mathbf{e}_\mu$. Note that $\mathbf{e}_{012} = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_0 \vee \mathbf{e}_1 \vee \mathbf{e}_2$. Let us see how this equation appears in terms of the original Clifford product in Euclidean spacetime. First we split the Dirac operator into temporal and space parts,

$$\nabla \vee \psi = \mathbf{e}_0 \vee \partial_0 \psi + \mathbf{e}_i \vee \partial_i \psi$$

Working on the temporal part, we have

$$\begin{aligned} \mathbf{e}_0 \vee \partial_0 \psi &= \partial_0 \psi \mathbf{e}_0 - 2(\partial_0 \psi \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{e}_0) \\ &= \partial_0 \psi \mathbf{e}_0 - 2\left[\frac{1}{2}(\partial_0 \psi \mathbf{e}_0 - \mathbf{e}_0 \partial_0 \psi)\right] = \mathbf{e}_0 \partial_0 \psi \end{aligned}$$

where we have used $\partial_0 \psi \cdot \mathbf{e}_0 = \frac{1}{2}(\partial_0 \psi \mathbf{e}_0 - \mathbf{e}_0 \partial_0 \psi)$ For the space part, we have

$$\mathbf{e}_i \vee \partial_i \psi = \partial_i \psi \mathbf{e}_i - 2[(\partial_i \psi) \cdot (\mathbf{e}_0 \cdot \mathbf{e}_i)] = \partial_i \psi \mathbf{e}_i$$

Therefore

$$\begin{aligned} \nabla \vee \psi &= \mathbf{e}_0 \partial_0 \psi + \partial_i \psi \mathbf{e}_i, & \nabla \vee \nabla \vee \psi &= \square_M \psi \\ \square_M &= \partial_0^2 - \sum_{i=1}^3 \partial_i^2, & \nabla^2 \psi &= \square_E \psi, & \square_E &= \partial_0^2 + \sum_{i=1}^3 \partial_i^2 \end{aligned}$$

The massive term in (13) will be

$$m\psi \vee \mathbf{e}_{012} = m\mathbf{e}_{12}\psi\mathbf{e}_0$$

Now we can write an equation simulating the Minkowskian Dirac equation in $Cl_{4,0}$,

$$\mathbf{e}_0 \partial_0 \psi + \partial_i \psi \mathbf{e}_i - m\mathbf{e}_{12}\psi\mathbf{e}_0 = 0 \quad (9)$$

If one consider a charged fermion field ψ in interaction with the electromagnetic field A , we will add to the Dirac equation (13) the term $eA\psi\gamma_{12}$, where $A \in \mathbb{R}^{1,3}$; applying the vee product, we let

$$A \vee \psi \vee \mathbf{e}_{12} = \mathbf{e}_{12}(\psi A - 2(\psi \cdot \mathbf{e}_0)A_0)$$

We have never changed the algebra—we have been working with $Cl_{4,0}$. The solutions of (7) are solutions of (9), too. For a general $\psi \in Cl_{1,3}^+$, $\forall x \in M$, we get in (7) eight coupled differential equations; transforming this original ψ solution to its $Cl_{4,0}^+$ version and checking it with (18), we recover exactly the same coupled system.

5. THE GENERAL CASE

Lounesto (1993) studied the case involving the opposite signatures $(+ - - -)$ and $(- + + +)$. For $a, b \in Cl_{1,3}$, the tilt transformation, based on the even–odd decomposition $Cl_{1,3} = Cl_{1,3}^+ \oplus Cl_{1,3}^-$ first emphasized by Clifford (1878), was defined as

$$\underbrace{ab}_{Cl_{1,3} \text{ product}} \rightarrow \underbrace{b_+ a_+ + b_+ a_- + b_- a_+ - b_- a_-}_{Cl_{3,1} \text{ product}}$$

where the subscripts minus and plus are the odd and even parts. We reinterpret this transformation as a new product \vee_t given by

$$A_l \vee_{t(3,1)} B_k = (-1)^{kl} B_k A_l$$

The meaning of this expression is that we are defining a mapping from an algebra Cl to its opposite algebra Cl^{opp} , and it is not difficult to show that $Cl_{p,q}^{opp} = Cl_{q,p}$. In order to consider the general case, we consider the vee and tilt products. There are some similarities

$$\text{vee product: } A_l \vee B_k = (-1)^{kl} [B_k A_l - 2(B_k \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot A_l)]$$

$$\text{tilt product: } A_l \vee B_k = (-1)^{kl} B_k A_l$$

It is interesting to see the purpose of the term with the temporal component \mathbf{e}_0 . The vee product simulates the change of signature as

$$\underbrace{(+ + + +)}_{Cl_{4,0}} \rightarrow \underbrace{(+ - - -)}_{Cl_{1,3}}$$

where these signs correspond to the square of the canonical basis elements. All the squares change except for the temporal component. If we set up the inverse problem $(+ - - -) \rightarrow (+ + + +)$, the general expression for the corresponding vee product will be the same. This is very welcome since, for example, we can get back the Euclidean Dirac equation from the Minkowskian one as in (9) just by using again the \vee product. The same holds in the opposite direction, of course. For the tilt product,

$$\underbrace{(+ - - -)}_{Cl_{1,3}} \rightarrow \underbrace{(- + + +)}_{Cl_{3,1}}$$

we change all the squares and now we do not need to subtract any temporal term. Again the opposite problem $(- + + +) \rightarrow (+ - - -)$ keeps the same form. For the most interesting cases in physics, we have

$$\begin{array}{ccc} (+ + + +) & \leftrightarrow & (+ - - -) \\ \updownarrow & & \updownarrow \\ (- - - -) & \leftrightarrow & (- + + +) \end{array}$$

The change of signature in the same row is done using the vee product and between rows of the same column using the tilt product. If we want to do the change in a *diagonal* way, i.e., $(+ + + +) \leftrightarrow (- + + +)$, then we will need to compose vee and tilt products. All these products can be extended to other signatures. For example,

$$\begin{array}{ccccccccc} (+ + + +) & \leftrightarrow & (+ - - -) & \leftrightarrow & (- - + +) & \leftrightarrow & (- - - +) & \leftrightarrow & (- - - -) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ (- - - -) & \leftrightarrow & (- + + +) & \leftrightarrow & (+ + - -) & \leftrightarrow & (+ + + -) & \leftrightarrow & (+ + + +) \end{array}$$

If we want to change the signature along the same row, we only need to know the canonical basis element \mathbf{e}_μ , which does not change its square; then

$$A_l \vee B_k = (-1)^{kl}[B_k A_l - 2(B_k \cdot \mathbf{e}_\mu)(\mathbf{e}^\mu \cdot A_l)]$$

where μ 's are *not* summed. For the change between rows in the same column,

$$A_l \vee_i B_k = (-1)^{kl} B_k A_l$$

Our scheme holds in any dimension.

6. OTHER APPLICATIONS

While in our discussion of the Dirac equation, we were interested in simulating Minkowskian properties in a Euclidean spacetime, in this section, we are interested in simulating Euclidean properties in Minkowski spacetime. Therefore in relation to the notation, we are interested in simulating the product of $Cl_{4,0}$ in terms of the generators $\{\gamma_\mu\}$ ($\mu = 0, 1, 2, 3$) of $Cl_{1,3}$.

6.1 The Hodge Star Operator

The Hodge star operator $*$ in Minkowski spacetime can be written using $Cl_{1,3}$ as

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad *\Phi = \tilde{\Phi} \gamma_5 \quad (11)$$

where $\Phi \in Cl_{1,3}$ is a multivector field and by the tilde we denoted the reversion operation such that

$$\tilde{A}_k = (-1)^{k(k-1)/2} A_k \in \Lambda^k$$

Using the vee product, we can write the Hodge star operator \star corresponding to Euclidean spacetime in terms of the algebra $Cl_{1,3}$ of Minkowski spacetime as

$$\star\Phi = \tilde{\Phi} \vee \gamma_5 \quad (12)$$

The Hodge operator in Euclidean spacetime is denoted by an asterisk, and that in Minkowski spacetime is denoted by a star. It is easy to see that $\gamma_0 \vee \gamma_1 \vee \gamma_2 \vee \gamma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ and that $\gamma_5 \vee \gamma_5 = 1$, while $\gamma_5 \gamma_5 = -1$. In order to rewrite the above expression using the definition of the vee product, it is convenient to split Φ into even and odd parts, that is, $\Phi = \Phi_+ + \Phi_-$, where $\Phi_\pm = \pm \hat{\Phi}_\pm$, where by the caret we denote the graded involution,

$$\hat{A}_k = (-1)_k A_k \in \Lambda^k$$

Now we can write, using the definition of the vee product, that

$$\begin{aligned} \Phi \vee \gamma_5 &= \Phi_+ \vee \gamma_5 + \Phi_- \vee \gamma_5 \\ &= \gamma_5 \hat{\Phi}_+ + 2\gamma_{123}(\gamma_0 \cdot \hat{\Phi}_+) + \gamma_5 \hat{\Phi}_- + 2\gamma_{123}(\gamma_0 \cdot \hat{\Phi}_-) \end{aligned} \quad (13)$$

$$\Phi_+ \vee \gamma_5 = \gamma_5 \gamma_0 \Phi_+ \gamma_0, \quad \Phi_- \vee \gamma_5 = -\gamma_5 \gamma_0 \Phi_- \gamma_0 \quad (14)$$

$$\Phi \vee \gamma_5 = \gamma_5 \gamma_0 \hat{\Phi} \gamma_0 \quad (15)$$

The Euclidean Hodge star operator \star can be written as

$$\star \Phi = \gamma_5 \gamma_0 \bar{\Phi} \gamma_0, \quad \bar{\Phi} = \hat{\Phi} = \check{\Phi} \quad (16)$$

The parity operation \mathcal{P} in $Cl_{1,3}$ is given by (Hestenes, 1966; Lounesto, 1993)

$$\mathcal{P}(\Phi) = \gamma_0 \Phi \gamma_0 \quad (17)$$

and if we rewrite (16) as

$$\star \Phi = -\gamma_0 \check{\Phi} \gamma_5 \gamma_0 = -\mathcal{P}(\star \Phi) \quad (18)$$

this expression shows that \star is written only in terms of operations on Minkowski spacetime.

6.2. Differential and Codifferential Operators

In Minkowski spacetime, the differential d and codifferential δ operators can be written in terms of Dirac operator ∇ as (Hestenes, 1966)

$$d\Phi = \frac{1}{2}(\nabla \Phi + \hat{\Phi} \overleftarrow{\nabla}), \quad \delta \Phi = \frac{1}{2}(\nabla \Phi - \hat{\Phi} \overleftarrow{\nabla}) \quad (19)$$

and such that $\nabla = d + \delta$, where $\nabla = \gamma^\mu \partial_\mu$ and the right action of the Dirac operator denoted by $\overleftarrow{\nabla}$ is defined as $\hat{\Phi} \overleftarrow{\nabla} = (\partial_\mu \Phi) \gamma^\mu$. With the above definition, we have that $\delta = *d*$, where $*$ is the Hodge star operator in Minkowski spacetime. One can find different definitions for the codifferential operator δ , but this is completely irrelevant for our purpose, which is to give examples of applications of our method.

Now we want to write the differential and codifferential operators corresponding to the case of Euclidean spacetime using the algebra of Minkowski spacetime. Let us denote these operators by \check{d} and $\check{\delta}$, respectively, the check being used to distinguish them from their counterparts in Minkowski spacetime. Since the differential operator is defined independently of the existence of a metric structure on a manifold, we have that $\check{d} = d$; on the other hand, the codifferential operator requires a metric for its definition and therefore we have $\check{\delta} \neq \delta$. In order to write the Euclidean version of these operators in Minkowski spacetime, the recipe is to replace the usual Clifford product by the vee product in formulas (19), that is,

$$\check{d}\Phi = \frac{1}{2}(\nabla \vee \Phi + \hat{\Phi} \vee \overleftarrow{\nabla}), \quad \check{\delta} \Phi = \frac{1}{2}(\nabla \vee \Phi - \hat{\Phi} \vee \overleftarrow{\nabla}) \quad (20)$$

The expressions $\nabla \vee \Phi = \gamma^\mu \vee \partial_\mu \Phi$ and $\Phi \vee \overleftarrow{\nabla} = \partial_\mu \Phi \vee \gamma^\mu$ can be calculated as before. We have for $\Phi_k \in \Lambda^k$ that ($i = 1, 2, 3$)

$$\nabla \vee \Phi_k = \gamma^0 \partial_0 \Phi + (-1)^k (\partial_i \Phi_k) \gamma^i, \quad \Phi_k \vee \overleftarrow{\nabla} = (-1)^k \gamma^i \partial_i \Phi_k + \partial_0 \Phi_k \gamma^0 \quad (21)$$

$$\nabla \vee \Phi = \gamma^0 \partial_0 \Phi + \partial_i \hat{\Phi} \gamma^i, \quad \Phi \vee \overleftarrow{\Phi} = \gamma^i \partial_i \hat{\Phi} + \partial_0 \Phi \gamma^0 \quad (22)$$

Now, using the expression for \check{d} given in (20), we see that

$$\begin{aligned} \check{d}\Phi &= \frac{1}{2}(\gamma^0 \partial_0 \Phi + \partial_i \hat{\Phi} \gamma^i + \gamma^i \partial_i \Phi + \partial_0 \hat{\Phi} \gamma^0) \\ &= \frac{1}{2}(\gamma^\mu \partial_\mu \Phi + \partial_\mu \hat{\Phi} \gamma^\mu) = d\Phi \end{aligned} \quad (23)$$

that is, $\check{d} = d$, as expected.

On the other hand, using the expression for $\check{\delta}$, we have that

$$\begin{aligned} \check{\delta}\Phi &= \frac{1}{2}(\gamma^0 \partial_0 \Phi + \partial_i \hat{\Phi} \gamma^i - \gamma^i \partial_i \Phi - \partial_0 \hat{\Phi} \gamma^0) \\ &= \frac{1}{2}[(\gamma^0 \partial_0 \Phi - \gamma^i \partial_i \Phi) - (\partial_0 \hat{\Phi} \gamma^0 - \partial_i \hat{\Phi} \gamma^i)] \end{aligned} \quad (24)$$

and clearly $\check{\delta} \neq \delta$, as expected. Using (16) for the Euclidean Hodge star operator \star written in Minkowski spacetime, we have that

$$d \star \Phi = \frac{1}{2}(-\gamma_5 \gamma^\mu \gamma_0 \partial_\mu \overline{\Phi} \gamma_0 + \gamma_5 \gamma_0 \partial_\mu \overline{\Phi} \gamma_0 \gamma^\mu) \quad (25)$$

$$\begin{aligned} \star d \star \Phi &= \frac{1}{2}(\gamma_5 \partial_\mu \overline{\Phi} \gamma_0 \gamma^\mu \gamma_5 \gamma_0 + \gamma_5 \gamma_0 \gamma^\mu \gamma_0 \partial_\mu \overline{\Phi} \gamma_5) \\ &= \frac{1}{2}(\gamma_0 \gamma^\mu \gamma_0 \partial_\mu \Phi - \partial_\mu \hat{\Phi} \gamma_0 \gamma^\mu \gamma_0) \\ &= \frac{1}{2}[(\gamma^0 \partial_0 \Phi - \gamma^i \partial_i \Phi) - (\partial_0 \hat{\Phi} \gamma^0 - \partial_i \hat{\Phi} \gamma^i)] \end{aligned} \quad (26)$$

and comparing this with (20), we have, as expected,

$$\check{\delta} = \star \check{d} \star = \star d \star \quad (27)$$

6.3. Self-Dual and Anti-Self-Dual Solutions of Gauge Field Equations

Since for what follows it is irrelevant whether or not we are considering Abelian or non-Abelian gauge fields, we shall consider the electromagnetic field—U(1) gauge fields—as an example for our discussion.

Let us consider the free Maxwell equations in Minkowski spacetime,

$$dF = 0, \quad \delta F = 0 \quad (28)$$

These equations do not have self-dual and/or anti-self-dual solutions, that is, solutions satisfying the conditions $F = \pm \star F$ in the *real* case. Such solutions are possible in Minkowski spacetime only if we *complexify* the underlying algebra considering a complex F . On the other hand, self-dual and/or anti-self-dual solutions of Maxwell equations exist in Euclidean spacetime without the need of any complexification. This can be described in our scheme by writing the equivalent Maxwell equations in Minkowski spacetime using the *vee* product.

The equations

$$\check{d}F = 0, \quad \check{\delta}F = 0 \quad (29)$$

are written in terms of the *real* Clifford algebra of Minkowski spacetime and admit self-dual and anti-self-dual solutions satisfying $F = \pm \star F$. This condition reads

$$F = \pm \gamma_5 \gamma_0 \bar{F} \gamma_0 = \pm \gamma_5 \gamma_0 \tilde{F} \gamma_0 \quad (30)$$

Let

$$F = E + \gamma_5 B, \quad E = \frac{1}{2}(F - \gamma_0 F \gamma_0), \quad \gamma_5 B = \frac{1}{2}(F + \gamma_0 F \gamma_0)$$

Then $E \gamma_0 = -\gamma_0 E$ and $\gamma_5 B \gamma_0 = \gamma_0 \gamma_5 B$, and (30) is satisfied for

$$E = \pm B$$

which are the usual self-dual and anti-self-dual solutions of the Euclidean case, but now written in terms of Minkowski spacetime.

7. CONCLUSIONS

Given the Clifford algebra of a quadratic space with a given signature, we have defined a new product in this structure such that it simulates the Clifford product of a quadratic space with another arbitrary signature different from the original one. We have used this in order to give an algebraic approach to the so-called Wick rotation. We have used this new product in order to simulate the product associated with the Minkowski spacetime in terms of the Clifford algebra of the Euclidean spacetime. We have also shown how to write the Minkowskian Dirac equation in Euclidean spacetime and in the other way how to write the Hodge star operator and the differential and codifferential operators corresponding to the Euclidean case in terms of Minkowski spacetime, discussing self-dual and anti-self-dual solutions for the gauge field equations in this case.

REFERENCES

- Baylis, William, Editor (1996). *Clifford (Geometric) Algebras with Applications in Physics, Mathematics and Engineering*, Birkhäuser, Boston.
- Chisholm, J. S. R., and A. K. Common, Editors (1986). *Clifford Algebras and Their Applications in Mathematical Physics*, Reidel, Dordrecht.
- Clifford, W. K. (1878). Applications of Grassmann's Extensive Algebra, *American Journal of Mathematics*, **1**, 350–358.
- Delanghe, R., F. Brackx, and H. Serras, Editors (1993). *Clifford Algebras and Their Applications in Mathematical Physics*, Kluwer, Dordrecht.

- Dietrich, V., K. Habetha, and G. Jank, Editors (1998). *Clifford Algebras and their Applications in Mathematical Physics*, Kluwer, Dordrecht.
- Figueiredo, V. L., Edmundo Capelas de Oliveira, and Waldyr Alves Rodrigues (1990). Covariant, algebraic and operator spinors, *International Journal of Theoretical Physics*, **29**, 371–395.
- Gilbert, J. E., and M. A. M. Murray (1991). *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge.
- Glimm, J., and A. Jaffe (1987). *Quantum Physics. A Functional Integral Point of View*, Springer-Verlag, New York.
- Hestenes, David (1966). *Space-Time Algebra*, Gordon and Breach, New York.
- Lounesto, Pertti (1993). Clifford algebras and Hestenes spinors, *Foundations of Physics*, **23**, 1203–1237.
- Lounesto, Pertti (1997). *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge.
- Micali, Artibano, R. Boudet, and J. Helmstetter, Editor (1992). *Clifford Algebras and Their Applications in Mathematical Physics*, Kluwer, Dordrecht.
- Parra, Josep Manel (1992). On Dirac and Dirac–Darwin–Hestenes equations, in: Artibano Micali, R. Boudet, and J. Helmstetter, Editors, *Clifford Algebras and Their Applications in Mathematical Physics*, Kluwer, Dordrecht, pp. 463–477.
- Pezzaglia, W. M., and J. J. Adams (1997). Should metric signature matter in Clifford algebra formulations of physical theories? e-Print Archive gr-qc/9704048.
- Porteous, Ian (1995). *Clifford Algebras and the Classical Groups*, Cambridge University Press, Cambridge.
- Riesz, Marcel (1993). *Clifford Numbers and Spinors*, Kluwer, Dordrecht.
- Rodrigues, Waldyr Alves, Quintino A. G. de Souza, Jayme Vaz, and Pertti Lounesto (1996). Dirac–Hestenes spinor fields on Riemann–Cartan manifolds, *International Journal of Theoretical Physics*, **35**, 1849–1900.
- van Nieuwenhuizen, P., and A. Waldron (1996). On Euclidean spinors and Wick rotations, *Physics Letters B*, **389**, 29–36.